M-math 2nd year Back paper Exam
Subject : Fourier Analysis

Time : 3.00 hours Max.Marks 45.

1. Let $\psi(x) := I_{[0,\frac{1}{2})}(x) - I_{[\frac{1}{2},1)}(x)$, where I_A is the indicator function of the set A. Let $\psi_{jk}(x) := 2^{\frac{j}{2}}\psi(2^j x - k), j, k \in \mathbb{Z}$. Show that $\{\psi_{jk}\}$ is an ortho-normal set in $L^2(\mathbb{R})$. (10)

2. a).Define the periodisation $k_t(x)$ of the heat kernel $p_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}, x \in \mathbb{R}^d$ and for $f \in L^1(T^d)$ compute the Fourier series of the convolution $f * k_t(x)$ in terms of $\hat{f}(n)$.
b) In a) does the Fourier series converge to $f * k_t(x)$? Justify your answer. (10 +5)

3. a) Let $f \in L^1(T^d)$ and extend f to $\mathbb{R}^d$ by $f(x + ke_i) = f(x), x \in T^d, k \in \mathbb{Z}$ and $e_i = (0, \cdots, 1, \cdots, 0), i = 1, \cdots, d$ are the basis vectors in $\mathbb{R}^d$. Show that $\int_{T^d} f(x-y)dx = \int_{T^d} f(x)\, dx$.
b) Does f belong to $L^1(\mathbb{R}^d)$? Prove your answer. (10 + 5)

4. Let f and ϕ be as in the Paley-Weiner theorem i.e.

$$f(z) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int \phi(t) e^{-iz\cdot t} dt, \quad z \in \mathbb{C}^d$$

where $\phi \in C_c^{\infty}(\mathbb{R}^d)$ and $supp(\phi) \subset B(0, r)$. Show that

$$|f(z)| \leq \gamma_N (1 + |z|)^{-N} e^{r|Im(z)|},$$

for $z \in \mathbb{C}^d, N \geq 1$ an integer and γ_N a constant depending on N. (10)

5. a) Let $0 \leq r < 1$ and $Q_r(\theta) := \frac{2r\sin\theta}{1+r^2-2r\cos\theta}, \theta \in (-\pi, \pi)$ be the conjugate Poisson kernel. Show that $Q_r(\theta) = -i \sum\limits_{n\in\mathbb{Z}} sgn(n) r^{|n|} e^{in\theta}$.
b) Show further that for $f \in L^2(-\pi, \pi), \lim\limits_{r\uparrow 1} Q_r f(\theta) = Hf(\theta)$, where $H : L^2 \to L^2$ is the Hilbert transform. (10)